

# Some Rarita-Schwinger Type Operators

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## Abstract

In this paper we study a generalization of the classical Rarita-Schwinger type operators and construct their fundamental solutions. We give some basic integral formulas related to these operators. We also establish that the projection operators appearing in the Rarita-Schwinger operators and the Rarita-Schwinger equations are conformally invariant. We further obtain the intertwining operators for other operators related to the Rarita-Schwinger operators under actions of the conformal group.

**Keywords:** Clifford algebra, Almansi-Fischer decomposition, conformal transformations, inner product.

**Classification:** Primary 30G35; Secondary 53C27

## 1 Introduction

In representation theory for  $O(n)$  and  $SO(n)$ , one considers functions  $f : U \rightarrow \mathcal{H}_k$  where  $U$  is a domain in  $\mathbb{R}^n$  and  $\mathcal{H}_k$  is the space of harmonic polynomials homogeneous of degree  $k$ . Such spaces are invariant under actions of  $O(n)$ . If one refines to the covering group  $Spin(n)$  of  $SO(n)$ , one replaces spaces of harmonic polynomials with spaces of homogeneous polynomial solutions to the Euclidean Dirac equation arising in Clifford analysis. See [BDS]. Clifford analysis is the study of and applications of Dirac type operators. In this context the Rarita-Schwinger operators arise. See [BSSV1, BSSV2, Va1, Va2, LRV1, LRV2, LR]. The Rarita-Schwinger operators are generalizations of the Dirac operator which in turn is a natural generalization of the Cauchy-Riemann operator. Rarita-Schwinger operators are also known as Stein-Weiss operators after [SW]. We denote a Rarita-Schwinger operator by  $R_k$ , where  $k = 0, 1, \dots, m, \dots$ . When  $k = 0$  we have the Dirac operator.

Here we start by constructing the Rarita-Schwinger operators and their fundamental solutions. This is based on the fundamental solution of the Dirac operator. Next, we give a summary of results on Rarita-Schwinger operators appearing in [BSSV1], giving detailed proofs and extending some of those results. We present a more detailed and alternative approach to that given in [BSSV1]. This includes a Stokes' Theorem, Borel-Pompeiu Formula, Cauchy's Integral Formula and a Cauchy Transform. We also obtain intertwining operator for  $R_k$  under actions of the conformal group, together with intertwining operators for the kernels to the Rarita-Schwinger operators, and the conformal invariance of Cauchy's Theorem and Cauchy's Integral Formula.

All of this ultimately helps to build the basics of Rarita-Schwinger type operators, including a theory of Rarita-Schwinger operators on examples of conformally flat manifolds. See for instance [LRV1, LRV2, LR].

## 2 Preliminaries

A Clifford algebra,  $Cl_n$ , can be generated from  $\mathbb{R}^n$  by considering the relationship

$$\underline{x}^2 = -\|\underline{x}\|^2$$

for each  $\underline{x} \in \mathbb{R}^n$ . We have  $\mathbb{R}^n \subseteq Cl_n$ . If  $e_1, \dots, e_n$  is an orthonormal basis for  $\mathbb{R}^n$ , then  $\underline{x}^2 = -\|\underline{x}\|^2$  tells us that  $e_i e_j + e_j e_i = -2\delta_{ij}$ . Let  $A = \{j_1, \dots, j_r\} \subset \{1, 2, \dots, n\}$  and  $1 \leq j_1 < j_2 < \dots < j_r \leq n$ . An arbitrary element of the basis of the Clifford algebra can be written as  $e_A = e_{j_1} \cdots e_{j_r}$ . Hence for any element  $a \in Cl_n$ , we have  $a = \sum_A a_A e_A$ , where  $a_A \in \mathbb{R}$ . For  $a \in Cl_n$ , we will need the following anti-involutions:

- Reversion:

$$\tilde{a} = \sum_A (-1)^{|A|(|A|-1)/2} a_A e_A,$$

where  $|A|$  is the cardinality of  $A$ . In particular,  $\widetilde{e_{j_1} \cdots e_{j_r}} = e_{j_r} \cdots e_{j_1}$ . Also  $\widetilde{\tilde{a}b} = \tilde{b}\tilde{a}$  for  $a, b \in Cl_n$ .

- Clifford conjugation:

$$\bar{a} = \sum_A (-1)^{|A|(|A|+1)/2} a_A e_A$$

satisfying  $\overline{e_{j_1} \cdots e_{j_r}} = (-1)^r e_{j_r} \cdots e_{j_1}$  and  $\overline{\bar{a}b} = \bar{b}\bar{a}$  for  $a, b \in Cl_n$ .

For each  $a = a_0 + \dots + a_{1\dots n} e_1 \cdots e_n \in Cl_n$  the scalar part of  $\bar{a}a$  gives the square of the norm of  $a$ , namely  $a_0^2 + \dots + a_{1\dots n}^2$ .

The Pin and Spin groups play an important role in Clifford analysis. The Pin group can be defined as

$$Pin(n) := \{a \in Cl_n : a = y_1 \dots y_p : y_1, \dots, y_p \in \mathbb{S}^{n-1}, p \in \mathbb{N}\}$$

and is clearly a group under multiplication in  $Cl_n$ .

Now suppose that  $y \in \mathbb{S}^{n-1} \subseteq \mathbb{R}^n$ . Look at  $xyy = yx^{\parallel y}y + yx^{\perp y}y = -x^{\parallel y} + x^{\perp y}$  where  $x^{\parallel y}$  is the projection of  $x$  onto  $y$  and  $x^{\perp y}$  is perpendicular to  $y$ . So  $xyy$  gives a reflection of  $x$  in the  $y$  direction. By the Cartan–Dieudonné Theorem each  $O \in O(n)$  is the composition of a finite number of reflections. If  $a = y_1 \dots y_p \in$

$Pin(n)$ , then  $\tilde{a} := y_p \dots y_1$  and  $ax\tilde{a} = O_a(x)$  for some  $O_a \in O(n)$ . Choosing  $y_1, \dots, y_p$  arbitrarily in  $\mathbb{S}^{n-1}$ , we see that the group homomorphism

$$\theta : Pin(n) \longrightarrow O(n) : a \longmapsto O_a$$

with  $a = y_1 \dots y_p$  and  $O_a(x) = ax\tilde{a}$  is surjective. Further  $-ax(-\tilde{a}) = ax\tilde{a}$ , so  $1, -1 \in \ker(\theta)$ . In fact  $\ker(\theta) = \{\pm 1\}$ . The Spin group is defined as

$$Spin(n) := \{a \in Pin(n) : a = y_1 \dots y_p \text{ and } p \text{ even}\}$$

and is a subgroup of  $Pin(n)$ . There is a group homomorphism

$$\theta : Spin(n) \longrightarrow SO(n)$$

which is surjective with kernel  $\{1, -1\}$ . See [P] for details.

The Dirac Operator in  $\mathbb{R}^n$  is defined to be

$$D := \sum_{j=1}^n e_j \frac{\partial}{\partial x_j}.$$

Note  $D^2 = -\Delta_n$ , where  $\Delta_n$  is the Laplacian in  $\mathbb{R}^n$ .

Let  $\mathcal{M}_k$  denote the space of  $Cl_n$ -valued polynomials, homogeneous of degree  $k$  and such that if  $p_k \in \mathcal{M}_k$  then  $Dp_k = 0$ . Such a polynomial is called a left monogenic polynomial homogeneous of degree  $k$ . Note if  $h_k \in \mathcal{H}_k$ , the space of  $Cl_n$ -valued harmonic polynomials homogeneous of degree  $k$ , then  $Dh_k \in \mathcal{M}_{k-1}$ . But  $Dup_{k-1}(u) = (-n - 2k + 2)p_{k-1}(u)$ , so

$$\mathcal{H}_k = \mathcal{M}_k \bigoplus u\mathcal{M}_{k-1}, h_k = p_k + up_{k-1}.$$

This is the so-called Almansi-Fischer decomposition of  $\mathcal{H}_k$ . See [BDS, R2].

Note that if  $Df(u) = 0$  then  $\bar{f}(u)\bar{D} = -\bar{f}(u)D = 0$ . So we can talk of right monogenic polynomials, homogeneous of degree  $k$  and we obtain by conjugation a right Almansi-Fisher decomposition,

$$\mathcal{H}_k = \overline{\mathcal{M}_k} \bigoplus \overline{\mathcal{M}_{k-1}}u,$$

where  $\overline{\mathcal{M}_k}$  stands for the space of right monogenic polynomials homogeneous of degree  $k$ .

### 3 The Rarita-Schwinger Operator $R_k$

Suppose  $U$  is a domain in  $\mathbb{R}^n$ . Consider a function of two variables

$$f : U \times \mathbb{R}^n \longrightarrow Cl_n$$

such that for each  $x \in U$ ,  $f(x, u)$  is a left monogenic polynomial homogeneous of degree  $k$  in  $u$ . Consider the action of the Dirac operator:

$$D_x f(x, u).$$

As  $Cl_n$  is not commutative then  $D_x f(x, u)$  is no longer monogenic in  $u$  but it is still harmonic and homogeneous of degree  $k$  in  $u$ . So by the Almansi-Fischer decomposition,  $D_x f(x, u) = f_{1,k}(x, u) + u f_{2,k-1}(x, u)$  where  $f_{1,k}(x, u)$  is a left monogenic polynomial homogeneous of degree  $k$  in  $u$  and  $f_{2,k-1}(x, u)$  is a left monogenic polynomial homogeneous of degree  $k-1$  in  $u$ . Let  $P_k$  be the left projection map

$$P_k : \mathcal{H}_k \rightarrow \mathcal{M}_k,$$

then  $R_k f(x, u)$  is defined to be  $P_k D_x f(x, u)$ . The left Rarita-Schwinger equation is defined to be (see [BSSV1])

$$R_k f(x, u) = 0.$$

We also have a right projection  $P_{k,r} : \mathcal{H}_k \rightarrow \overline{\mathcal{M}_k}$ , and a right Rarita-Schwinger equation  $f(x, u) D_x P_{k,r} = f(x, u) R_k = 0$ . Since

$$D_x f(x, u) = p_k(x, u) + u p_{k-1}(x, u) \quad \text{and} \quad D_u u p_{k-1}(x, u) = -(n+2k-2) p_{k-1}(x, u),$$

we have  $u p_{k-1}(x, u) = -\frac{1}{n+2k-2} u D_u D_x f(x, u)$ . Thus  $(1 - P_k) D_x f(x, u) = u p_{k-1}(x, u) = -\frac{1}{n+2k-2} u D_u D_x f(x, u)$ . Hence

$$P_k D_x f(x, u) = \frac{1}{n+2k-2} u D_u D_x f(x, u) + D_x f(x, u) = \left( \frac{u D_u}{n+2k-2} + 1 \right) D_x f(x, u).$$

So we obtain that  $P_k = \left( \frac{u D_u}{n+2k-2} + 1 \right)$  and  $R_k = \left( \frac{u D_u}{n+2k-2} + 1 \right) D_x$ . See [BSSV1].

It is crucial to ask if there are any non-trivial solutions to this equation. First for any  $k$ -monogenic polynomial  $p_k(u)$  we have trivially  $R_k p_k(u) = 0$ . In particular the reproducing kernel of  $\mathcal{M}_k$  is annihilated by  $R_k$ . We now produce a representation of this reproducing kernel. Consider the fundamental solution  $G(u) = \frac{1}{\omega_n} \frac{-u}{\|u\|^n}$  to the Dirac operator  $D$ , where  $\omega_n$  is the surface area of the unit sphere,  $\mathbb{S}^{n-1}$ .

Consider the Taylor series expansion of  $G(v - u)$  and restrict to the  $k$ th order terms in  $u_1, \dots, u_n$  ( $u = u_1 e_1 + \dots + u_n e_n$ ). These terms have as vector valued coefficients

$$\frac{\partial^k}{\partial v_1^{k_1} \dots \partial v_n^{k_n}} G(v) \quad (k_1 + \dots + k_n = k).$$

As  $G(v)$  is a solution to the Dirac equation,  $DG(v) = \sum_{i=1}^n e_j \frac{\partial G(v)}{\partial v_j} = 0$ , we can replace  $\frac{\partial}{\partial v_1}$  by  $-\sum_{j=2}^n e_1^{-1} e_j \frac{\partial}{\partial v_j}$ . Doing this each time  $\frac{\partial}{\partial v_1}$  occurs and collecting like terms we obtain a finite series of polynomials homogeneous of degree  $k$  in  $u$

$$\sum_{\sigma} P_{\sigma}(u) V_{\sigma}(v)$$

where the summation is taken over all permutations of monogenic polynomials  $(u_2 - u_1 e_1^{-1} e_2), \dots, (u_n - u_1 e_1^{-1} e_n)$ , each term in the summation contains  $j_2$  copies of  $(u_2 - u_1 e_1^{-1} e_2), \dots, j_n$  copies of  $(u_n - u_1 e_1^{-1} e_n)$ , and

$$P_{\sigma}(u) = \frac{1}{k!} \Sigma(u_{i_1} - u_1 e_1^{-1} e_{i_1}) \dots (u_{i_k} - u_1 e_1^{-1} e_{i_k}), V_{\sigma}(v) = \frac{\partial^k G(v)}{\partial v_2^{j_2} \dots \partial v_n^{j_n}}$$

$j_2 + \dots + j_n = k$ , and  $i_k \in \{2, \dots, n\}$ . Here summation is taken over all permutations of the monomials without repetition. See [BDS]. Note that this series is the sum of the  $k$ -th order terms in the Taylor expansion of  $G(v - u)$  and consequently it is a vector.

Now  $\int_{\mathbb{S}^{n-1}} V_{\sigma}(u) u P_{\mu}(u) dS(u) = \delta_{\sigma, \mu}$  where  $\delta_{\sigma, \mu}$  is the Kronecker delta and  $\mu$  is a set of  $n - 1$  non-negative integers summing to  $k$ . See [BDS]. Following [BDS] it can be seen that the polynomial  $P_{\sigma}$  is left monogenic and the set of all such polynomials, homogeneous of degree  $k$ , forms a basis for the right  $Cl_n$  module  $\mathcal{M}_k$ . Consequently, the expression

$$Z_k(u, v) := \sum_{\sigma} P_{\sigma}(u) V_{\sigma}(v) v$$

is the reproducing kernel of  $\mathcal{M}_k$  with respect to integration over  $\mathbb{S}^{n-1}$  (see [BDS]). Further as  $Z_k(u, v)$  does not depend on  $x$ ,

$$R_k Z_k = 0.$$

Note that  $V_{\sigma}(v) v$  on  $\mathbb{S}^{n-1}$  extends to  $V_{\sigma}(-v^{-1}) G(v)$  on  $\mathbb{R}^n$  and this function is a right monogenic polynomial in  $v$  and it is homogeneous of degree  $k$ . See [BDS] and elsewhere.

We may ask if there are any solutions to  $R_k f = 0$  that depend on  $x$ . To do this we look at the interaction of the operator  $R_k$  with conformal transformations.

## 4 Conformal transformations

We first establish the invariance properties of the projection operator  $P_k$ .

## 4.1 Conformal invariance of the projection $P_k$

Let  $P_{k,w}$  and  $P_{k,u}$  be the projections with respect to  $w$  and  $u$  respectively.

### 4.1.1 Orthogonal transformations

Let  $x = ay\tilde{a}$ , and  $u = aw\tilde{a}$ .

**Lemma 1.**  $P_{k,w}\tilde{a}f(ay\tilde{a}, aw\tilde{a}) = \tilde{a}P_{k,u}f(x, u)$ , where  $a \in Pin(n)$ .

**Proof** Let  $f(x, u) = f_1(x, u) + uf_2(x, u)$ , where  $f_1(x, u)$  and  $f_2(x, u)$  are monogenic polynomials homogeneous of degree  $k$  and  $k - 1$  in  $u$ . So  $P_{k,u}f(x, u) = f_1(x, u) = f_1(ay\tilde{a}, aw\tilde{a})$  and  $\tilde{a}f(ay\tilde{a}, aw\tilde{a}) = \tilde{a}f_1(ay\tilde{a}, aw\tilde{a}) + \tilde{a}aw\tilde{a}f_2(ay\tilde{a}, aw\tilde{a}) = \tilde{a}f_1(ay\tilde{a}, aw\tilde{a}) \pm w\tilde{a}f_2(ay\tilde{a}, aw\tilde{a})$ .

Further as  $\tilde{a}f_1(ay\tilde{a}, aw\tilde{a})$  and  $\tilde{a}f_2(ay\tilde{a}, aw\tilde{a})$  are monogenic polynomials homogeneous of degree  $k$  and  $k - 1$  in  $w$  respectively, we have  $P_{k,w}\tilde{a}f(ay\tilde{a}, aw\tilde{a}) = \tilde{a}f_1(ay\tilde{a}, aw\tilde{a}) = \tilde{a}P_{k,u}f(x, u)$ . ■

### 4.1.2 Inversion

Let  $x = y^{-1}$ ,  $u = \frac{ywy}{\|y\|^2}$ .

**Lemma 2.**  $P_{k,w}\frac{y}{\|y\|^n}f(y^{-1}, \frac{ywy}{\|y\|^2}) = \frac{y}{\|y\|^n}P_{k,u}f(x, u)$ .

**Proof** Since  $f(x, u) = f_1(x, u) + uf_2(x, u)$ , by substitution we have

$$f(y^{-1}, \frac{ywy}{\|y\|^2}) = f_1(y^{-1}, \frac{ywy}{\|y\|^2}) + \frac{ywy}{\|y\|^2}f_2(y^{-1}, \frac{ywy}{\|y\|^2}).$$

Now multiplying both sides of the above equation by  $\frac{y}{\|y\|^n}$ , one gets

$$\begin{aligned} \frac{y}{\|y\|^n}f(y^{-1}, \frac{ywy}{\|y\|^2}) &= \frac{y}{\|y\|^n}f_1(y^{-1}, \frac{ywy}{\|y\|^2}) + \frac{y}{\|y\|^n}\frac{ywy}{\|y\|^2}f_2(y^{-1}, \frac{ywy}{\|y\|^2}) \\ &= \frac{y}{\|y\|^n}f_1(y^{-1}, \frac{ywy}{\|y\|^2}) - w\frac{y}{\|y\|^n}f_2(y^{-1}, \frac{ywy}{\|y\|^2}). \end{aligned}$$

Now Let  $P_{k,w}$  act on the previous equation. We have

$$P_{k,w}\frac{y}{\|y\|^n}f(y^{-1}, \frac{ywy}{\|y\|^2}) = \frac{y}{\|y\|^n}f_1(y^{-1}, \frac{ywy}{\|y\|^2}) = \frac{y}{\|y\|^n}f_1(x, u) = \frac{y}{\|y\|^n}P_{k,u}f(x, u),$$

which follows from the facts that  $\frac{y}{\|y\|^n}f_1(y^{-1}, \frac{ywy}{\|y\|^2})$  and  $\frac{y}{\|y\|^n}f_2(y^{-1}, \frac{ywy}{\|y\|^2})$  are monogenic and homogeneous of degree  $k$  and  $k - 1$  in  $w$ . ■

### 4.1.3 Translations

Let  $x = y + a, a \in \mathbb{R}^n$ . In order to keep the homogeneity of  $f(x, u)$  in  $u$ ,  $u$  does not change under translation. So we have

**Lemma 3.**  $P_k f(x, u) = P_k f(y + a, u)$ , where  $x = y + a$ .

### 4.1.4 Dilations

Let  $x = \lambda y$ , where  $\lambda \in \mathbb{R}^+$ . It is obvious to observe that  $P_k$  is invariant under dilation.

**Lemma 4.**  $P_k f(x, u) = P_k f(\lambda y, u)$ , where  $x = \lambda y$ .

Hence  $P_k$  is conformally invariant.

Ahlfors [A] and Vahlen [V] show that given a Möbius transformation  $y = \phi(x)$  on  $\mathbb{R}^n \cup \{\infty\}$  it can be expressed as  $y = (ax + b)(cx + d)^{-1}$  where  $a, b, c, d \in Cl_n$  and satisfy the following conditions:

- i.  $a, b, c, d$  are all products of vectors in  $\mathbb{R}^n$ .
- ii.  $a\tilde{b}, c\tilde{d}, \tilde{b}c, \tilde{d}a \in \mathbb{R}^n$ .
- iii.  $a\tilde{d} - b\tilde{c} = \pm 1$ .

When  $c = 0, \phi(x) = (ax + b)(cx + d)^{-1} = axd^{-1} + bd^{-1} = \pm ax\tilde{a} + bd^{-1}$ . Now assume  $c \neq 0$ , then  $\phi(x) = (ax + b)(cx + d)^{-1} = ac^{-1} \pm (cx\tilde{c} + d\tilde{c})^{-1}$ , this is the so-called Iwasawa decomposition. Using this notation and the conformal weights,

$f(\phi(x))$  is changed to  $J(\phi, x)f(\phi(x))$ , where  $J(\phi, x) = \frac{\widetilde{cx + d}}{\|cx + d\|^n}$ . Note when  $\phi(x) = x + a$  then  $J(\phi, x) \equiv 1$ . Now using the Iwasawa decomposition, we get the following result:

**Theorem 1.**  $P_{k,w} J(\phi, x) f(\phi(x), \frac{\widetilde{(cx + d)w(cx + d)}}{\|cx + d\|^2}) = J(\phi, x) P_{k,u} f(\phi(x), u)$ ,

where  $u = \frac{\widetilde{(cx + d)w(cx + d)}}{\|cx + d\|^2}$  and where  $P_{k,w}$  and  $P_{k,u}$  are the projections with respect to  $w$  and  $u$  respectively.

Note that if the Möbius transformation is either translation or dilation then  $u = w$ . This explains why in Lemma 4 the term  $u$  is not multiplied by  $\lambda$ .

Lemmas 1 and 2 and Theorem 1 establish intertwining relationships for the projection operator,  $P_k$ , under actions of the conformal group.

## 4.2 Conformal invariance of the Rarita-Schwinger operator $R_k$

Now let us establish the intertwining operators for  $R_k$  and the conformal invariance of the equation  $R_k f = 0$ . Let  $R_{k,u}$  and  $R_{k,w}$  be the Rarita-Schwinger operators with respect to  $u$  and  $w$  respectively.

We will need the following. If we have the Möbius transformation  $y = \phi(x)$  and  $D_x$  is the Dirac operator with respect to  $x$  and  $D_y$  is the Dirac operator with respect to  $y$  then  $D_x = J_{-1}(\phi, x)^{-1} D_y J_1(\phi, x)$ , where  $J_{-1}(\phi, x) = \frac{cx + d}{\|cx + d\|^{n+2}}$

and  $J_1(\phi, x) = J(\phi, x) = \frac{\widetilde{cx + d}}{\|cx + d\|^n}$ . See [R1].

### 4.2.1 Orthogonal transformations $O \in O(n), a \in Pin(n)$

**Theorem 2.** *If  $x = ay\tilde{a}, u = aw\tilde{a}$ , then  $aR_{k,u}f(x, u) = R_{k,w}\tilde{a}f(ay\tilde{a}, aw\tilde{a})$ .*

**Proof**

$$R_{k,u}f(x, u) = P_{k,u}D_x f(x, u) = P_{k,u}a^{-1}D_y \tilde{a}f(ay\tilde{a}, u)$$

Therefore, by Lemma 1

$$aP_{k,u}a^{-1}D_y \tilde{a}f(ay\tilde{a}, u) = P_{k,w}aa^{-1}D_y \tilde{a}f(ay\tilde{a}, aw\tilde{a}) = R_{k,w}\tilde{a}f(ay\tilde{a}, aw\tilde{a}). \quad \blacksquare$$

In fact, Theorem 2 tells us that if  $R_k f(x, u) = 0$  then  $R_k \tilde{a}f(ay\tilde{a}, aw\tilde{a}) = 0$ .

### 4.2.2 Inversion

Let  $x = y^{-1}, (= \frac{-y}{\|y\|^2})$ .

**Theorem 3.** *Set  $u = \frac{ywy}{\|y\|^2}$ , then  $\frac{y}{\|y\|^{n+2}}R_{k,u}f(x, u) = R_{k,w}G(y)f(y^{-1}, \frac{ywy}{\|y\|^2})$ .*

**Proof**

$$R_{k,u}f(x, u) = P_{k,u}D_x f(x, u) = P_{k,u}G_{-1}(y)^{-1}D_y G(y)f(y^{-1}, u),$$

where  $G_{-1}(y) = y\|y\|^n$ .

Therefore by Lemma 2,

$$\begin{aligned} & G_{-1}(y)P_{k,u}G_{-1}(y)^{-1}D_y G(y)f(y^{-1}, u) \\ &= P_{k,w}G_{-1}(y)G_{-1}(y)^{-1}D_y G(y)f(y^{-1}, \frac{ywy}{\|y\|^2}) = R_{k,w}G(y)f(y^{-1}, \frac{ywy}{\|y\|^2}). \quad \blacksquare \end{aligned}$$

Consequently, if  $R_k f(x, u) = 0$ , then  $R_k G(y)f(y^{-1}, \frac{ywy}{\|y\|^2}) = 0$ .



### 4.2.3 Dilations

Let  $x = \lambda y$ ,  $\lambda \in \mathbb{R}^+$ .  $R_k f(x, u) = R_k f(\lambda y, u)$  and if  $R_k f(x, u) = 0$  then  $R_k f(\lambda y, u) = 0$ .

### 4.2.4 Translations

Let  $x = y + a$ ,  $a \in \mathbb{R}^n$ . In order to preserve homogeneity of polynomials in  $u$ ,  $f(x, u)$  is transformed under a translation by  $a$  to  $f(y + a, u)$  (Note: otherwise the action of the Vahlen matrices is not correct). So  $R_k f(x, u) = R_k f(y + a, u)$  and  $R_k f(x, u) = 0$  implies  $R_k f(y + a, u) = 0$ , where  $x = y + a$ .

Now using the Iwasawa decomposition of  $(ax + b)(cx + d)^{-1}$ , we obtain intertwining operators for  $R_k$  :

**Theorem 4.**

$$R_{k,x,w} J_1(\phi, x) \psi(\phi(x), \frac{\widetilde{(cx + d)w(cx + d)}}{\|cx + d\|^2}) = J_{-1}(\phi, x) R_{k,y,u} \psi(y, u),$$

$$\text{where } y = \phi(x), u = \frac{\widetilde{(cx + d)w(cx + d)}}{\|cx + d\|^2}, R_{k,x,w} = P_{k,w} D_x \text{ and } R_{k,y,u} = P_{k,u} D_y.$$

Consequently, we obtain that  $R_k f(x, u) = 0$  implies

$$R_k J(\phi, x) f(\phi(x), \frac{\widetilde{(cx + d)w(cx + d)}}{\|cx + d\|^2}) = 0,$$

where  $u = \frac{\widetilde{(cx + d)w(cx + d)}}{\|cx + d\|^2}$ . For this last formula see also [BSSV1].

## 5 A Kernel for $R_k$ and Some Basic Integral Formulas

Now applying inversion from the left we obtain that if  $R_k Z_k(u, v) = 0$  then  $R_k G(x) Z_k(\frac{xux}{\|x\|^2}, v) = 0$ . That is,

$$F_k(x, u, v) = \frac{x}{\|x\|^n} Z_k(\frac{xux}{\|x\|^2}, v) = \frac{x}{\|x\|^{n+2k}} Z_k(xux, v)$$

is a non-trivial solution to  $R_k f(x, u) = 0$  on  $\mathbb{R}^n \setminus \{0\}$ . Note that this function is monogenic in  $u$ .

Similarly, applying inversion from the right we obtain that

$$Z_k(u, \frac{xx}{\|x\|^2}) \frac{x}{\|x\|^n} = Z_k(u, xvx) \frac{x}{\|x\|^{n+2k}}$$

is a non-trivial solution to  $f(x, v)R_k = 0$  on  $\mathbb{R}^n \setminus \{0\}$ . In fact, [BSSV1], this function is  $F_k(x, u, v)$ , and, up to a constant,  $F_k(x, u, v)$  is the fundamental solution of  $R_k$ . One proof of this statement is given in the following. Let  $\mathcal{M}_k$  denote the space of left monogenic polynomials homogeneous of degree  $k$  and suppose  $a \in Pin(n)$ , then

$$l_{\tilde{a}} : \mathcal{M}_k \rightarrow \mathcal{M}_k \quad : f(u) \rightarrow \tilde{a}f(au\tilde{a})$$

is an isomorphism. Similarly, let  $\overline{\mathcal{M}}_k$  denote the space of right monogenic polynomials homogeneous of degree  $k$  then

$$r_{\tilde{a}} : \overline{\mathcal{M}}_k \rightarrow \overline{\mathcal{M}}_k \quad : f(u) \rightarrow f(au\tilde{a})a$$

is also an isomorphism. Using these isomorphisms it may be seen that for each  $a \in Pin(n)$  then  $\pm \tilde{a}Z_k(au\tilde{a}, av\tilde{a})a$  is also the reproducing kernel for  $\mathcal{M}_k$ . We choose the plus sign when  $a \in Spin(n)$  and the minus sign when  $a \in Pin(n) \setminus Spin(n)$ . Now consider a non-zero vector  $x \in \mathbb{R}^n$ , then  $\frac{x}{\|x\|} \in Pin(n)$ . So we have

$$Z_k(u, v) = -\frac{x}{\|x\|} Z_k(\frac{xx}{\|x\|^2}, \frac{xx}{\|x\|^2}) \frac{x}{\|x\|},$$

that is,  $-\frac{x}{\|x\|} Z_k(\frac{xx}{\|x\|^2}, \frac{xx}{\|x\|^2}) \frac{x}{\|x\|}$  is also the reproducing kernel for  $\mathcal{M}_k$ . Now we look at the fundamental solution of  $R_k$  which has the representation  $Z_k(u, \frac{xx}{\|x\|^2}) \frac{x}{\|x\|^n}$ . Then using the previous equality, we get

$$Z_k(u, \frac{xx}{\|x\|^2}) \frac{x}{\|x\|^n} = -\frac{x}{\|x\|} Z_k(\frac{xx}{\|x\|^2}, v) \frac{x}{\|x\|} \frac{x}{\|x\|^n} = \frac{x}{\|x\|^n} Z_k(\frac{xx}{\|x\|^2}, v).$$

Further suppose  $\mu$  is a  $Cl_n$  valued measure on  $\mathbb{R}^n$  with compact support,  $[\mu]$ . It follows for suitable choices of  $\mu$  the integral  $\int_{[\mu]} F_k(x, u, v) d\mu$  defines a solution to  $R_k f = 0$  on  $(\mathbb{R}^n \setminus [\mu]) \times \mathbb{R}^n$ .

## 5.1 Stokes' Theorem

We first build Stokes' Theorem for the Rarita-Schwinger operator. This is based on Stokes' Theorem for the Dirac operator.

**Theorem 5.** (*Stokes' Theorem for the Dirac operator, [BDS]*)

Let  $\Omega$  and  $\Omega'$  be domains in  $\mathbb{R}^n$  and suppose the closure of  $\Omega$  lies in  $\Omega'$ . Further

suppose the closure of  $\Omega$  is compact and  $\partial\Omega$  is piecewise smooth. Let  $f, g \in C^1(\Omega', Cl_n)$ . Then

$$\int_{\partial\Omega} g(x, u) d\sigma_x f(x, u) = \int_{\Omega} [(g(x, u) D_x) f(x, u) + g(x, u) (D_x f(x, u))] dx^n,$$

where  $dx^n = dx_1 \wedge \cdots \wedge dx_n$ ,  $d\sigma_x = n(x) d\sigma(x)$ ,  $\sigma$  is scalar Lebesgue measure on  $\partial\Omega$  and  $n(x)$  is unit outer normal vector to  $\partial\Omega$ . We may write  $n(x)$  as  $\sum_{i=1}^n n_i(x) e_i$ , where  $n_i(x)$  are scalar-valued functions.  $g(x, u) D_x$  means  $D_x$  acts from the right on  $g(u, x)$ .

**Definition 1.** For any  $Cl_n$ -valued polynomials  $P(u), Q(u)$ , the inner product  $(P(u), Q(u))_u$  with respect to  $u$  is given by

$$(P(u), Q(u))_u = \int_{\mathbb{S}^{n-1}} P(u) Q(u) dS(u).$$

This inner product differs slightly from the Fischer inner product in [BSSV1]. There the inner product is  $\int_{\mathbb{S}^{n-1}} \overline{R}(u) Q(u) dS(u)$  for a  $Cl_n$  valued polynomial  $R(u)$ . If we place  $R(u) = \overline{P}(u)$  we see that, as the conjugation<sup>-</sup> is an isomorphism, the two inner products are equivalent. For any  $p_k \in \mathcal{M}_k$ , one obtains (see [BDS])

$$p_k(u) = (Z_k(u, v), p_k(v))_v = \int_{\mathbb{S}^{n-1}} Z_k(u, v) p_k(v) dS(v).$$

Using Stokes' Theorem for the Dirac operator, we can obtain the basic formulas related to the Rarita-Schwinger operators.

**Lemma 5.** Suppose  $p_k$  is a left monogenic polynomial homogeneous of degree  $k$  and  $p_{k-1}$  is a left monogenic polynomial homogeneous of degree  $k-1$  then

$$\int_{\mathbb{S}^{n-1}} \tilde{p}_{k-1}(u) u p_k(u) dS(u) = 0.$$

Outline Proof: As we are integrating over the unit sphere the previous integral can be written as

$$\int_{\mathbb{S}^{n-1}} \tilde{p}_{k-1}(u) n(u) p_k(u) dS(u).$$

By the Clifford-Cauchy Theorem [BDS] this integral vanishes. ■

We now have

**Theorem 6.** (*Rarita-Schwinger Stokes' Theorem*) [BSSV1] Let  $\Omega'$  and  $\Omega$  be as in Theorem 5. Then for  $f, g \in C^1(\Omega', \mathcal{M}_k)$ , we have

$$\begin{aligned} & \int_{\partial\Omega} (g(x, u) d\sigma_x f(x, u))_u \\ &= \int_{\Omega} (g(x, u) R_k, f(x, u))_u dx^n + \int_{\Omega} (g(x, u), R_k f(x, u))_u dx^n. \end{aligned}$$

Further

$$\begin{aligned} & \int_{\partial\Omega} (g(x, u) d\sigma_x f(x, u))_u \\ &= \int_{\partial\Omega} (g(x, u), P_k d\sigma_x f(x, u))_u \\ &= \int_{\partial\Omega} (g(x, u) d\sigma_x P_k, f(x, u))_u. \end{aligned}$$

Outline Proof: The first identity is obtained by first applying Stokes' Theorem to the integral  $\int_{\partial\Omega} (g(x, u) d\sigma_x f(x, u))_u$  to obtain

$$\int_{\Omega} (g(x, u) D, f(x, u))_u + (g(x, u), D f(x, u))_u dx^n.$$

Both  $g(x, u) D$  and  $D f(x, u)$  have an Almansi-Fischer decomposition with respect to  $u$ . So applying Lemma 5 with respect to  $u$  and Definition 1 and these Almansi-Fischer decompositions give the result.

The second collection of identities again arise by applying the Almansi-Fischer decomposition  $d\sigma_x f(x, u)$  and  $g(x, u) d\sigma_x$  respectively, and then applying Definition 1 and Lemma 5 with respect to  $u$ . ■

Now if both  $f(x, u)$  and  $g(x, u)$  are solutions of  $R_k$ , then we have the following result:

**Corollary 1.** (*Cauchy's Theorem*)

If  $R_k f(x, u) = 0$  and  $g(x, u) R_k = 0$  for  $f, g \in C^1(\Omega', \mathcal{M}_k)$ , then

$$\int_{\partial\Omega} (g(x, u), P_k d\sigma_x f(x, u))_u = 0.$$

Let  $S$  be a hypersurface in  $\mathbb{R}^n$  and  $y = \phi(x) = (ax + b)(cx + d)^{-1}$ . Now look at Cauchy's Theorem:

$$\begin{aligned} 0 &= \int_S (g(y, u), P_k d\sigma_y f(y, u))_u = \int_S (g(y, u), P_k n(y) f(y, u))_u d\sigma(y) \\ &= \int_{\phi^{-1}(S)} \left( g(\phi(x), u), P_k \tilde{J}(\phi, x) n(x) J(\phi, x) f(\phi(x), u) \right)_u d\sigma(x) \\ &= \int_{\phi^{-1}(S)} \int_{\mathbb{S}^{n-1}} g(\phi(x), u) P_{k,u} \tilde{J}(\phi, x) n(x) J(\phi, x) f(\phi(x), u) dS(u) d\sigma(x). \end{aligned}$$

Set  $u = \frac{\widetilde{(cx+d)w(cx+d)}}{\|cx+d\|^2}$ , since  $P_{k,u}$  can interchange with  $\tilde{J}(\phi, x)$ , the previous equation equals

$$\begin{aligned}
0 &= \int_{\phi^{-1}(S)} \int_{\mathbb{S}^{n-1}} g(\phi(x), \frac{\widetilde{(cx+d)w(cx+d)}}{\|cx+d\|^2}) \tilde{J}(\phi, x) P_{k,w} n(x) J(\phi, x) f(\phi(x), \\
&\quad \frac{\widetilde{(cx+d)w(cx+d)}}{\|cx+d\|^2}) dS(w) d\sigma(x) \\
&= \int_{\phi^{-1}(S)} (g(\phi(x), \frac{\widetilde{(cx+d)w(cx+d)}}{\|cx+d\|^2}) \tilde{J}(\phi, x), P_{k,w} d\sigma_x J(\phi, x) \\
&\quad f(\phi(x), \frac{\widetilde{(cx+d)w(cx+d)}}{\|cx+d\|^2}))_w.
\end{aligned}$$

Therefore, Cauchy's Theorem is conformally invariant under Möbius transformations.

We now wish to introduce the Borel-Pompeiu Theorem from [BSSV1]. First we will need:

**Lemma 6.** *Suppose  $h_k : \mathbb{R}^n \rightarrow Cl_n$  is a harmonic polynomial homogeneous of degree  $k$  and  $n > 2$ . Suppose  $u \in \mathbb{S}^{n-1}$  then*

$$\frac{1}{\omega_n} \int_{\mathbb{S}^{n-1}} h_k(xux) dS(x) = c_k h_k(u),$$

where  $c_k = \frac{n-2}{n-2+2k}$ .

The proof follows from [DX] (Proposition 2.2.3 on page 34).

**Proof** Fix  $u \in \mathbb{S}^{n-1}$ , suppose  $h_k : \mathbb{R}^n \rightarrow Cl_n$  is a harmonic polynomial homogeneous of degree  $k$ . Now compute

$$I(h_k) = \frac{2}{\omega_n} \int_{\langle x, u \rangle > 0} h_k(u - 2 \langle x, u \rangle x) dS(x),$$

note  $xux = u - 2 \langle x, u \rangle x$  is the reflection of  $u$  in the mirror  $x^\perp$  (where  $\int_{\langle x, u \rangle > 0} dS(x) = \frac{1}{2} \int_{\mathbb{S}^{n-1}} dS(x) = \frac{\omega_n}{2}$ , and  $\|x\| = 1$ ). Since  $xux = u - 2 \langle x, u \rangle x$  is invariant under  $x \rightarrow -x$  we have

$$I(h_k) = \frac{1}{\omega_n} \int_{\mathbb{S}^{n-1}} h_k(u - 2 \langle x, u \rangle x) dS(x).$$

By rotation invariance assume  $u = (0, \dots, 0, 1)$ . Any harmonic homogeneous polynomial of degree  $k$  has an expression

$$h_k(y) = \sum_{j=0}^k \|y\|^{k-j} P_{k-j}^{j+n/2-1} \left( \frac{y_n}{\|y\|} \right) h_{k_j}(y_1, \dots, y_{n-1}),$$

where  $h_{k_j}$  is harmonic and homogeneous of degree  $j$  and the normalized Gegenbauer polynomial is

$$P_m^\lambda(t) = \sum_{i=0}^m \frac{(-m)_i (m+2\lambda)_i}{(\lambda + \frac{1}{2})_i i!} \left( \frac{1-t}{2} \right)^i.$$

In the coordinate system  $y = (y' \sin \theta, \cos \theta)$  with  $0 \leq \theta \leq \pi$  and  $y' \in \mathbb{S}^{n-2}$  the integral is

$$\int_{\mathbb{S}^{n-1}} h_k(y) dS(x) = c'_n \int_0^\pi \sin^{n-2} \theta d\theta \int_{\mathbb{S}^{n-2}} h_k(y' \sin \theta, \cos \theta) dS_{n-2}(y').$$

Set  $u = (0, \dots, 0, 1)$  and  $x = (x' \sin \theta, \cos \theta)$  with  $x' \in \mathbb{S}^{n-2}$ , then

$$u - 2 \langle x, u \rangle x = (-2x' \cos \theta \sin \theta, 1 - 2 \cos^2 \theta).$$

Thus

$$\begin{aligned} & \int_{\mathbb{S}^{n-1}} h_k(u - 2 \langle x, u \rangle x) dS(x) \\ &= \sum_{j=0}^k c'_n \int_0^\pi P_{k-j}^{j+n/2-1} (1 - 2 \cos^2 \theta) \sin^{n-2} \theta d\theta \int_{\mathbb{S}^{n-2}} h_{k_j}(-2x' \cos \theta \sin \theta) dS_{n-2}(x') \\ &= \sum_{j=0}^k c'_n \int_0^\pi P_{k-j}^{j+n/2-1} (1 - 2 \cos^2 \theta) (-2 \cos \theta \sin \theta)^j \sin^{n-2} \theta d\theta \int_{\mathbb{S}^{n-2}} h_{k_j}(x') dS_{n-2}(x') \\ &= h_{k_0} c'_n \int_0^\pi P_k^{n/2-1} (1 - 2 \cos^2 \theta) \sin^{n-2} \theta d\theta. \end{aligned}$$

The integrals equal zero for  $j > 0$ , by the orthogonality property of harmonics.

Next set  $t = \cos \theta$  and  $dt = -\sin \theta d\theta$ , then

$$\begin{aligned} \int_0^\pi P_k^\lambda (1 - 2 \cos^2 \theta) \sin^{n-2} \theta d\theta &= \int_{-1}^1 P_k^\lambda (1 - 2t^2) (1 - t^2)^{\lambda - \frac{1}{2}} dt \\ &= \sum_{i=0}^k \frac{(-k)_i (k+2\lambda)_i}{(\lambda + \frac{1}{2})_i i!} \int_{-1}^1 t^{2i} (1 - t^2)^{\lambda - \frac{1}{2}} dt \\ &= \sum_{i=0}^k \frac{(-k)_i (k+2\lambda)_i}{(\lambda + \frac{1}{2})_i i!} B(i + \frac{1}{2}, \lambda + \frac{1}{2}) \\ &= \frac{\Gamma(\frac{1}{2}) \Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda + 1)} \sum_{i=0}^k \frac{(-k)_i (k+2\lambda)_i (\frac{1}{2})_i}{(\lambda + \frac{1}{2})_i i! (\lambda + 1)_i}. \end{aligned}$$

The Saalschütz summation formula (for  $-k + a + b + 1 = c + d$ ) is

$${}_3F_2 \left( \begin{matrix} -k, a, b \\ c, d \end{matrix}; 1 \right) = \frac{(c-a)_k (d-a)_k}{(c)_k (d)_k}.$$

$$\begin{aligned} \int_0^\pi P_k^\lambda (1 - 2 \cos^2 \theta) \sin^{n-2} \theta d\theta &= \frac{\Gamma(\frac{1}{2}) \Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda + 1)} \frac{(\lambda)_k (\lambda + \frac{1}{2})_k}{(\lambda + \frac{1}{2})_k (\lambda + 1)_k} \\ &= \frac{\Gamma(\frac{1}{2}) \Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda + 1)} \frac{\lambda}{\lambda + k}. \end{aligned}$$

The normalizing constant is (now  $\lambda = \frac{n}{2}$ )  $c'_n = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{1}{2}) \Gamma(\frac{n-1}{2})}$  and

$$\frac{1}{\omega_n} \int_{\mathbb{S}^{n-1}} h_k(u - 2 \langle x, u \rangle x) dS(x) = \frac{\frac{n}{2} - 1}{\frac{n}{2} - 1 + k} h_{k_0} = c_k h_{k_0},$$

and  $h_k(0, 0, \dots, 1) = h_{k_0}$ , where  $c_k = \frac{n-2}{n-2+2k}$ .  $\blacksquare$

We now introduce  $E_k(x, u, v) := \frac{1}{\omega_n c_k} F_k(x, u, v)$ .

**Theorem 7.** [BSSV1] (*Borel-Pompeiu Theorem*) Let  $\Omega'$  and  $\Omega$  be as in Theorem 5 and  $y \in \Omega$ . Then for  $f \in C^1(\Omega', \mathcal{M}_k)$

$$f(y, u) = \int_{\partial\Omega} (E_k(x - y, u, v), P_{k,v} d\sigma_x f(x, v))_v - \int_{\Omega} (E_k(x - y, u, v), R_{k,v} f(x, v))_v dx^n.$$

**Proof** Here we will use the representation

$$E_k(x - y, u, v) = \frac{1}{\omega_n c_k} Z_k(u, \frac{(x - y)v(x - y)}{\|x - y\|^2}) \frac{x - y}{\|x - y\|^n},$$

and  $R_{k,v}$  is the Rarita-Schwinger operator with respect to  $v$ . Consider a ball  $B(y, r)$  centered at  $y$  with radius  $r$  such that  $\overline{B(y, r)} \subset \Omega$ . By Stokes' Theorem, we have

$$\begin{aligned} &\int_{\Omega} (E_k(x - y, u, v), R_{k,v} f(x, v))_v dx^n \\ &= \int_{\Omega \setminus B(y, r)} (E_k(x - y, u, v), R_{k,v} f(x, v))_v dx^n + \int_{B(y, r)} (E_k(x - y, u, v), R_{k,v} f(x, v))_v dx^n. \end{aligned}$$

The second integral tends to zero as  $r$  tends to zero. This follows from the degree of homogeneity of  $E_k(x - y, u, v)$ . Now applying Stokes' Theorem to the first

integral, one gets

$$\begin{aligned} & \int_{\Omega \setminus B(y,r)} (E_k(x-y, u, v), R_{k,v}f(x, v))_v dx^n \\ &= \int_{\partial\Omega} (E_k(x-y, u, v), P_{k,v}d\sigma_x f(x, v))_v - \int_{\partial B(y,r)} (E_k(x-y, u, v), P_{k,v}d\sigma_x f(x, v))_v. \end{aligned}$$

Now let us look at

$$\begin{aligned} \int_{\partial B(y,r)} (E_k(x-y, u, v), P_{k,v}d\sigma_x f(x, v))_v dx^n &= \int_{\partial B(y,r)} (E_k(x-y, u, v), P_{k,v}d\sigma_x f(y, v))_v \\ &+ \int_{\partial B(y,r)} (E_k(x-y, u, v), P_{k,v}d\sigma_x [f(x, v) - f(y, v)])_v. \end{aligned}$$

Since the second integral on the right hand side tends to zero as  $r$  goes to zero, we only need to deal with the first integral.

$$\begin{aligned} & \int_{\partial B(y,r)} (E_k(x-y, u, v), P_{k,v}d\sigma_x f(y, v))_v \\ &= \int_{\partial B(y,r)} \int_{\mathbb{S}^{n-1}} E_k(x-y, u, v) P_{k,v} d\sigma_x f(y, v) dS(v) \\ &= \int_{\partial B(y,r)} \int_{\mathbb{S}^{n-1}} \frac{1}{\omega_n c_k} Z_k \left( u, \frac{(x-y)v(x-y)}{\|x-y\|^2} \right) \frac{x-y}{\|x-y\|^n} P_{k,v} n(x) f(y, v) dS(v) d\sigma(x), \end{aligned}$$

where  $n(x)$  is the unit outer normal vector and  $d\sigma(x)$  is the scalar measure on  $\partial B(y, r)$ . Now  $n(x)$  here is  $\frac{y-x}{\|x-y\|}$ . Hence the previous integral becomes

$$\frac{1}{\omega_n c_k} \int_{\partial B(y,r)} \int_{\mathbb{S}^{n-1}} Z_k \left( u, \frac{(x-y)v(x-y)}{\|x-y\|^2} \right) \frac{x-y}{\|x-y\|^n} P_{k,v} \frac{y-x}{\|x-y\|} f(y, v) dS(v) d\sigma(x).$$

By equation (1) this integral becomes

$$\begin{aligned} & \frac{1}{\omega_n c_k} \int_{\partial B(y,r)} \int_{\mathbb{S}^{n-1}} Z_k \left( u, \frac{(x-y)v(x-y)}{\|x-y\|^2} \right) \frac{x-y}{\|x-y\|^n} \frac{y-x}{\|x-y\|} f(y, v) dS(v) d\sigma(x) \\ &= \frac{1}{\omega_n} c_k^{-1} \int_{\partial B(y,r)} \frac{1}{r^{n-1}} \int_{\mathbb{S}^{n-1}} Z_k \left( u, \frac{(x-y)v(x-y)}{\|x-y\|^2} \right) f(y, v) d\sigma(x) dS(v) \end{aligned}$$

By Lemma 6 this integral is equal to

$$\int_{\mathbb{S}^{n-1}} Z_k(u, v) f(y, v) dS(v) = f(y, u). \quad \blacksquare$$



**Theorem 8.** [BSSV1](Cauchy's Integral Formula) If  $R_k f(x, v) = 0$ , then for  $y \in \Omega$ ,

$$\begin{aligned} f(y, u) &= \int_{\partial\Omega} (E_k(x - y, u, v), P_k d\sigma_x f(x, v))_v \\ &= \int_{\partial\Omega} (E_k(x - y, u, v) d\sigma_x P_{k,r} f(x, v))_v. \quad \blacksquare \end{aligned}$$

We now show the conformal invariance of Cauchy's Integral Formula. We start with inversion.

Since

$$\begin{aligned} x^{-1} - y^{-1} &= -y^{-1}(x - y)x^{-1} = -x^{-1}(x - y)y^{-1} \\ &= \frac{-x}{\|x\|^2}(x - y) \frac{y}{\|y\|^2} = \frac{-y}{\|y\|^2}(x - y) \frac{x}{\|x\|^2}, \end{aligned}$$

$$\begin{aligned} E_k(x^{-1} - y^{-1}, u, v) &= G(x^{-1} - y^{-1}) Z_k \left( \frac{(x^{-1} - y^{-1})u(x^{-1} - y^{-1})}{\|x^{-1} - y^{-1}\|^2}, v \right) \\ &= -G(y)^{-1} G(x - y) G(x)^{-1} Z_k \left( \frac{x(x - y)yuy(x - y)x}{\|x\|^2 \|y\|^2 \|x - y\|^2}, v \right) \\ &= -G(y)^{-1} G(x - y) G(x)^{-1} \frac{-x}{\|x\|} Z_k \left( \frac{(x - y)yuy(x - y)}{\|y\|^2 \|x - y\|^2}, \frac{vx}{\|x\|^2} \right) \frac{x}{\|x\|} \\ &= G(y)^{-1} G(x - y) Z_k \left( \frac{(x - y)u'(x - y)}{\|x - y\|^2}, \frac{vx}{\|x\|^2} \right) x \|x\|^{n-2}, \quad \text{set } u' = \frac{yuy}{\|y\|^2} \\ &= -G(y)^{-1} G(x - y) Z_k \left( \frac{(x - y)u'(x - y)}{\|x - y\|^2}, \frac{vx}{\|x\|^2} \right) G(x)^{-1} \\ &= -G(y)^{-1} E_k(x - y, u', v') G(x)^{-1}, \end{aligned}$$

where  $u' = \frac{yuy}{\|y\|^2}$  and  $v' = \frac{vx}{\|x\|^2}$ .

Now consider

$$\begin{aligned} E_k(ax\tilde{a} - ay\tilde{a}, u, v) &= G(a(x - y)\tilde{a}) Z_k \left( \frac{a(x - y)\tilde{a}ua(x - y)\tilde{a}}{\|a(x - y)\tilde{a}\|^2}, v \right) \\ &= aG(x - y)\tilde{a} Z_k \left( \frac{a(x - y)\tilde{a}ua(x - y)\tilde{a}}{\|x - y\|^2}, v \right) \\ &= \pm aG(x - y)\tilde{a} Z_k \left( \frac{\tilde{a}a(x - y)\tilde{a}ua(x - y)\tilde{a}a}{\|x - y\|^2}, \tilde{a}va \right) \tilde{a} \end{aligned}$$

$$= aG(x-y)Z_k \left( \frac{(x-y)u'(x-y)}{\|x-y\|^2}, \tilde{a}va \right) \tilde{a}, \quad \text{set } u' = \tilde{a}ua$$

$$= aE_k(x-y, u', v')\tilde{a},$$

where  $u' = \tilde{a}ua$  and  $v' = \tilde{a}va$ .

Using the Iwasawa decomposition, one gets

$$E_k(\phi(x) - \phi(y), u, v) = J(\phi, y)^{-1}E_k(x-y, u', v')\tilde{J}(\phi, x)^{-1},$$

where  $u' = \frac{\widetilde{(cy+d)u(cy+d)}}{\|cy+d\|^2}$ ,  $v' = \frac{\widetilde{(cx+d)v(cx+d)}}{\|cx+d\|^2}$ , and  $\phi$  is the Möbius transformation.

Suppose  $S$  is a smooth hypersurface lying in  $\mathbb{R}^n$ . Let  $x' = \phi(x)$  and  $y' = \phi(y)$ , now let us consider Cauchy's Integral Formula

$$f(y', u) = \int_S (E_k(x' - y', u, v), P_k n(x') f(x', v))_v d\sigma(x')$$

$$= \int_S \int_{\mathbb{S}^{n-1}} E_k(x' - y', u, v) P_k n(x') f(x', v) dS(v) d\sigma(x').$$

Thus, by the fact that  $n(x') d\sigma(x') = \tilde{J}(\phi, x) n(x) J(\phi, x) d\sigma(x)$  we have

$$f(\phi(y), u) = \int_{\phi^{-1}(S)} \int_{\mathbb{S}^{n-1}} J(\phi, y)^{-1} E_k(x-y, u', v') \tilde{J}(\phi, x)^{-1} P_{k,v} \tilde{J}(\phi, x) n(x) J(\phi, x) f(\phi(x), v) dS(v) d\sigma(x)$$

$$= \int_{\phi^{-1}(S)} \int_{\mathbb{S}^{n-1}} J(\phi, y)^{-1} E_k(x-y, u', v') \tilde{J}(\phi, x)^{-1} \tilde{J}(\phi, x) P_{k,v} n(x) J(\phi, x)$$

$$f(\phi(x), \frac{(cx+d)v'(\widetilde{cx+d})}{\|cx+d\|^2}) dS(v') d\sigma(x)$$

$$= \int_{\phi^{-1}(S)} \int_{\mathbb{S}^{n-1}} J(\phi, y)^{-1} E_k(x-y, u', v') P_{k,v} n(x) J(\phi, x) f(\phi(x), \frac{(cx+d)v'(\widetilde{cx+d})}{\|cx+d\|^2})$$

$$dS(v') d\sigma(x).$$

Multiplying both sides of the previous equation by  $J(\phi, y)$ , we obtain

$$J(\phi, y) f(\phi(y), \frac{(cy+d)u'(\widetilde{cy+d})}{\|cy+d\|^2}) = \int_{\phi^{-1}(S)} \int_{\mathbb{S}^{n-1}} E_k(x-y, u', v') P_{k,v} n(x) J(\phi, x)$$

$$f(\phi(x), \frac{(cx+d)v'(\widetilde{cx+d})}{\|cx+d\|^2}) dS(v') d\sigma(x),$$

where  $u = \frac{(cy + d)u'(\widetilde{cy + d})}{\|cy + d\|^2}$  and  $v = \frac{(cx + d)v'(\widetilde{cx + d})}{\|cx + d\|^2}$ .

Therefore, Cauchy's Integral Formula is conformally invariant.

Now if the function  $\psi$  has compact support in  $\Omega$ , then by the Borel-Pompeiu Theorem we have the following formula:

**Theorem 9.**  $\iint_{\mathbb{R}^n} -(E_k(x - y, u, v), R_k\psi(x, v))_v dx^n = \psi(y, u)$  for each  $\psi \in C_0^\infty(\mathbb{R}^n, \mathcal{M}_k)$ .

Similarly, we get a Cauchy transform for the Rarita-Schwinger operator  $R_k$  :

**Definition 2.** For a domain  $\Omega \subset \mathbb{R}^n$  and a function  $f : \Omega \times \mathbb{R}^n \longrightarrow Cl_n$ , where  $f(x, u)$  is monogenic in  $u$ , the Cauchy (or  $T_k$ -transform) of  $f$  is formally defined to be

$$(T_k f)(y, v) = - \iint_{\Omega} (E_k(x - y, u, v), f(x, u))_u dx^n, \quad y \in \Omega.$$

**Theorem 10.**  $R_k T_k \psi = \psi$  for  $\psi \in C_0^\infty(\mathbb{R}^n, \mathcal{M}_k)$ . i.e

$$R_k \iint_{\mathbb{R}^n} (E_k(x - y, u, v), \psi(x, u))_u dx^n = \psi(y, v).$$

**Proof** For each fixed  $y \in \mathbb{R}^n$ , let  $R(y)$  be a bounded rectangle in  $\mathbb{R}^n$  centered at  $y$ . Then

$$R_k \iint_{\mathbb{R}^n \setminus R(y)} (E_k(x - y, u, v), \psi(x, u))_u dx^n = 0.$$

Now consider

$$\begin{aligned} & \frac{\partial}{\partial y_i} \iint_{R(y)} (E_k(x - y, u, v), \psi(x, u))_u dx^n \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \iint_{R(y)} (E_k(x - y, u, v) - E_k(x - y - \varepsilon e_i, u, v), \psi(x, u))_u dx^n \end{aligned}$$

If we translate the rectangle by  $\varepsilon$  in  $-e_i$  direction, then the derivative will be shifted from  $E_k$  to  $\psi$ . Hence the previous integral becomes

$$\begin{aligned} & \iint_{R(y)} (E_k(x - y, u, v), \frac{\psi(x, u) - \psi(x + \varepsilon e_i)}{\varepsilon})_u dx^n + \\ & \frac{1}{\varepsilon} \iint_{(R(y + \varepsilon e_i) \setminus R(y)) \cup (R(y) \setminus R(y + \varepsilon e_i))} (E_k(x - y, u, v), \psi(x, u) - \psi(x + \varepsilon e_i, u))_u dx^n \end{aligned}$$

When  $\varepsilon$  tends to zero, the integral is equal to

$$\iint_{R(y)} (E_k(x-y, u, v), \frac{\partial \psi(x, u)}{\partial x_i})_u dx^n + \int_{\partial R_1(y) \cup \partial R_2(y)} (E_k(x-y, u, v), \psi(x, u))_u d\sigma(x)$$

where  $\partial R_1(y)$  and  $\partial R_2(y)$  are the two faces of  $R(y)$  with normal vectors  $\pm e_i$ . So

$$\begin{aligned} & D_y \iint_{R(y)} (E_k(x-y, u, v), \psi(x, u))_u dx^n \\ &= \iint_{R(y)} \sum_{i=1}^n e_i (E_k(x-y, u, v), \frac{\partial \psi(x, u)}{\partial x_i})_u dx^n \\ &+ \int_{\partial R(y)} n(x) (E_k(x-y, u, v), \psi(x, u))_u d\sigma(x). \end{aligned}$$

When the volume of  $R(y)$  tends to zero, the first integral tends to zero by the homogeneity of the kernel  $E_k$ . So we shall concentrate attention on the integral

$$P_k \int_{\partial R(y)} n(x) (E_k(x-y, u, v), \psi(x, u))_u d\sigma(x).$$

This is equal to

$$P_k \int_{\partial R(y)} \int_{\mathbb{S}^{n-1}} n(x) E_k(x-y, u, v) \psi(x, u) dS(u) d\sigma(x),$$

which in turn is equal to

$$\begin{aligned} & P_k \int_{\partial R(y)} \int_{\mathbb{S}^{n-1}} n(x) E_k(x-y, u, v) \psi(y, u) dS(u) d\sigma(x) \\ &+ P_k \int_{\partial R(y)} \int_{\mathbb{S}^{n-1}} n(x) E_k(x-y, u, v) (\psi(x, u) - \psi(y, u)) dS(u) d\sigma(x). \end{aligned}$$

But the last integral on the right side of the above formula tends to zero as the surface area of  $\partial R(y)$  tends to zero. Hence we are left with

$$P_k \int_{\partial R(y)} \int_{\mathbb{S}^{n-1}} n(x) E_k(x-y, u, v) \psi(y, u) dS(u) d\sigma(x).$$

By Stokes' Theorem this is equal to

$$P_k \int_{\partial B(y, r)} \int_{\mathbb{S}^{n-1}} n(x) E_k(x-y, u, v) \psi(y, u) dS(u) d\sigma(x).$$

In turn this is equal to

$$\begin{aligned} & P_k \int_{\partial B(y,r)} \frac{1}{\omega_n c_k} \int_{\mathbb{S}^{n-1}} \frac{y-x}{\|x-y\|} \frac{x-y}{\|x-y\|^n} Z_k\left(\frac{(x-y)u(x-y)}{\|x-y\|^2}, v\right) \psi(y, u) dS(u) d\sigma(x) \\ &= P_k \int_{\partial B(y,r)} \frac{1}{\omega_n c_k} \int_{\mathbb{S}^{n-1}} \frac{1}{r^{n-1}} Z_k\left(\frac{(x-y)u(x-y)}{\|x-y\|^2}, v\right) \psi(y, u) dS(u) d\sigma(x). \end{aligned}$$

By Lemma 6, the integral becomes  $P_k \int_{\mathbb{S}^{n-1}} Z_k(u, v) \psi(y, u) dS(u) = P_k \psi(y, v) = \psi(y, v)$ . ■

Now we may establish the intertwining operators for the convolution operator  $E_k \star$ . More precisely we shall show that:

**Theorem 11.** *If  $\psi \in C_0^\infty(\mathbb{R}^n, \mathcal{M}_k)$ , then*

$$\begin{aligned} & J_1(\phi, y) \iint_{\mathbb{R}^n} (E_k(x' - y', u, v), \psi(x', v))_v d(x')^n \\ &= \iint_{\mathbb{R}^n} (E_k(x - y, u', w) \tilde{J}_{-1}(\phi, x), \psi(\phi(x), w))_w dx^n, \end{aligned}$$

where  $x' = \phi(x)$ ,  $y' = \phi(y)$ ,  $u = \frac{(cy + d)u'(\widetilde{cy + d})}{\|cy + d\|^2}$  and  $v = \frac{(cx + d)w(\widetilde{cx + d})}{\|cx + d\|^2}$ .

Alternatively,

$$J_1(\phi, -) E_k \star \psi = E_k \tilde{J}_{-1}(\phi, -) \star \psi.$$

**Proof** First consider inversion, let  $\phi(x) = x^{-1}$ ,  $\phi(y) = y^{-1}$ . Then

$$\begin{aligned} & \iint_{\mathbb{R}^n} (E_k(x^{-1} - y^{-1}, u, v), \psi(x^{-1}, v))_v d(x^{-1})^n \\ &= \iint_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} E_k(x^{-1} - y^{-1}, u, v) \psi(x^{-1}, v) dS(v) d(x^{-1})^n \\ &= \iint_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} Z_k\left(u, \frac{(x^{-1} - y^{-1})v(x^{-1} - y^{-1})}{\|x^{-1} - y^{-1}\|^2}\right) \frac{x^{-1} - y^{-1}}{\|x^{-1} - y^{-1}\|^n} \psi(x^{-1}, v) dS(v) d(x^{-1})^n. \end{aligned}$$

Since

$$x^{-1} - y^{-1} = -y^{-1}(x - y)x^{-1} = -x^{-1}(x - y)y^{-1} = \frac{-x}{\|x\|^2}(x - y) \frac{y}{\|y\|^2} = \frac{-y}{\|y\|^2}(x - y) \frac{x}{\|x\|^2},$$

$$\begin{aligned} & Z_k\left(u, \frac{(x^{-1} - y^{-1})v(x^{-1} - y^{-1})}{\|x^{-1} - y^{-1}\|^2}\right) = Z_k\left(u, \frac{y(x - y)xv x(x - y)y}{\|x\|^2 \|y\|^2 \|x - y\|^2}\right) \\ &= -\frac{y}{\|y\|} Z_k\left(\frac{yuy}{\|y\|^2}, \frac{(x - y)w(x - y)}{\|x - y\|^2}\right) \frac{y}{\|y\|}, \quad \text{set } w = \frac{xvx}{\|x\|^2}. \end{aligned}$$

Now the previous integral becomes

$$\begin{aligned}
& \iint_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} \frac{y}{\|y\|} Z_k \left( \frac{yuy}{\|y\|^2}, \frac{(x-y)w(x-y)}{\|x-y\|^2} \right) \frac{y}{\|y\|} y \|y\|^{n-2} G(x-y) x \|x\|^{n-2} \\
& \quad \psi(\phi(x), v) \frac{1}{\|x\|^{2n}} dS(v) dx^n \\
&= \iint_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} -y \|y\|^{n-2} Z_k \left( \frac{yuy}{\|y\|^2}, \frac{(x-y)w(x-y)}{\|x-y\|^2} \right) G(x-y) \frac{x}{\|x\|^{n+2}} \psi(\phi(x), v) dS(v) dx^n \\
&= \iint_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} -y \|y\|^{n-2} E_k(x-y, u', w) \frac{x}{\|x\|^{n+2}} \psi(\phi(x), v) dS(v) dx^n,
\end{aligned}$$

where  $u' = \frac{yuy}{\|y\|^2}$ ,  $w = \frac{xvx}{\|x\|^2}$ . Then the previous integral is

$$\iint_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} -y \|y\|^{n-2} E_k(x-y, u', w) \frac{x}{\|x\|^{n+2}} \psi(\phi(x), \frac{xwx}{\|x\|^2}) dS(w) dx^n$$

Now multiplying both sides of the equation by  $\frac{y^{-1}}{\|y\|^{n-2}}$ , we obtain

$$\begin{aligned}
& \frac{y}{\|y\|^n} \iint_{\mathbb{R}^n} (E_k(x^{-1} - y^{-1}, u, v), \psi(x^{-1}, v))_v d(x^{-1})^n \\
&= \iint_{\mathbb{R}^n} \left( E_k(x-y, u', w) \frac{x}{\|x\|^{n+2}}, \psi(\phi(x), \frac{xwx}{\|x\|^2}) \right)_w dx^n,
\end{aligned}$$

where  $u = \frac{yu'y}{\|y\|^2}$  and  $v = \frac{xwx}{\|x\|^2}$ .

Next, consider orthogonal transformations. We will apply similar arguments used to establish the equation under inversion. Let  $\phi(x) = ax\tilde{a}$  and  $\phi(y) = ay\tilde{a}$ , where  $a \in Pin(n)$ . Then

$$\begin{aligned}
& \iint_{\mathbb{R}^n} (E_k(ax\tilde{a} - ay\tilde{a}, u, v), \psi(\phi(x), v))_v d(ax\tilde{a})^n \\
&= \iint_{\mathbb{R}^n} (E_k(a(x-y)\tilde{a}, u, v), \psi(\phi(x), v))_v d(ax\tilde{a})^n \\
&= \iint_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} Z_k(u, \frac{a(x-y)\tilde{a}va(x-y)\tilde{a}}{\|x-y\|^2}) \frac{a(x-y)\tilde{a}}{\|x-y\|^n} \psi(\phi(x), v) dS(v) dx^n \\
&= \pm \iint_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} a Z_k(\tilde{a}ua, \frac{(x-y)\tilde{a}va(x-y)}{\|x-y\|^2}) \tilde{a} \frac{a(x-y)\tilde{a}}{\|x-y\|^n} \psi(\phi(x), v) dS(v) dx^n.
\end{aligned}$$

Set  $w = \tilde{a}va$ , then  $v = aw\tilde{a}$ . Hence the integral becomes

$$\begin{aligned} & \iint_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} aZ_k(\tilde{a}ua, \frac{(x-y)w(x-y)}{\|x-y\|^2}) \frac{(x-y)\tilde{a}}{\|x-y\|^n} \psi(\phi(x), v) dS(v) dx^n \\ &= \iint_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} aE_k(x-y, u', w) \tilde{a} \psi(\phi(x), aw\tilde{a}) dS(w) dx^n, \end{aligned}$$

where  $u' = \tilde{a}ua$ . Now multiplying both sides of the equation by  $a^{-1}$ , we have

$$\begin{aligned} & \tilde{a} \iint_{\mathbb{R}^n} (E_k(ax\tilde{a} - ay\tilde{a}, u, v), \psi(\phi(x), v))_v d(ax\tilde{a})^n \\ &= \iint_{\mathbb{R}^n} E_k(x-y, u', w) \tilde{a} \psi(\phi(x), aw\tilde{a}) dS(w) dx^n, \end{aligned}$$

where  $u' = \tilde{a}ua$  and  $v = aw\tilde{a}$ . By the Iwasawa decomposition of  $\phi(x) = (ax + b)(cx + d)^{-1}$ , we obtain

$$\begin{aligned} & J_1(\phi, y) \iint_{\mathbb{R}^n} (E_k(x' - y', u, v), \psi(x', v))_v d(x')^n \\ &= \iint_{\mathbb{R}^n} (E_k(x-y, u', w) \tilde{J}_{-1}(\phi, x), \psi(\phi(x), w))_w dx^n, \end{aligned}$$

where  $J_1(\phi, x) = J(\phi, x) = \frac{\widetilde{cx+d}}{\|cx+d\|^n}$ ,  $J_{-1}(\phi, x) = \frac{cx+d}{\|cx+d\|^{n+2}}$ ,  $x' = \phi(x)$ ,  $y' = \phi(y)$ ,  $u = \frac{(cy+d)u'(\widetilde{cy+d})}{\|cy+d\|^2}$  and  $v = \frac{(cx+d)w(\widetilde{cx+d})}{\|cx+d\|^2}$ . Alternatively,

$$J_1(\phi, -)E_k \star \psi = E_k \tilde{J}_{-1}(\phi, -) \star \psi. \quad \blacksquare$$

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